Tutorial 7

March 16, 2017

1. (a) Find the Fourier sine series of $\phi(x) = x$ on the interval $[0, l]$.
   (b) Find the Fourier cosine series of $\phi(x) = x$ on the interval $[0, l]$.
   (c) Find the full Fourier series of $\phi(x) = x$ on the interval $[-l, l]$.

   Solution: (a)The Fourier sine series of $\phi(x) = x$ is

$$\phi(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$$

where the coefficients are

$$\begin{aligned} B_n =& \frac{2}{l} \int_0^l x \sin(\frac{n\pi x}{l}) dx \\ =& -\frac{2x}{n\pi} \cos(\frac{n\pi x}{l}) \Big|_0^l + \frac{2}{n\pi} \int_0^l \cos(\frac{n\pi x}{l}) dx \\ =& (-1)^{n+1} \frac{2l}{n\pi}, \quad n = 1, 2, \cdots \end{aligned}$$

Hence

$$x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2l}{n\pi} \sin \frac{n\pi x}{l} = \frac{2l}{\pi} (\sin \frac{\pi x}{l} - \frac{1}{2} \sin \frac{2\pi x}{l} + \frac{1}{3} \sin \frac{3\pi x}{l} - \cdots).$$

(b)The Fourier cosine series of $\phi(x) = x$ is

$$\phi(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l}$$

where the coefficients are

$$A_0 = \frac{2}{l} \int_0^l x dx = l,$$

and

$$\begin{aligned} A_n =& \frac{2}{l} \int_0^l x \cos(\frac{n\pi x}{l}) dx \\ =& \frac{2x}{n\pi} \sin(\frac{n\pi x}{l}) \Big|_0^l - \frac{2}{n\pi} \int_0^l \sin(\frac{n\pi x}{l}) dx \\ =& \frac{2l}{n^2\pi^2} \cos(\frac{n\pi x}{l}) \Big|_0^l = \frac{2l}{n^2\pi^2} ((-1)^n - 1), \quad n = 1, 2, \cdots \end{aligned}$$

Hence

$$x = \frac{l}{2} - \sum_{n=1, n \text{ odd}}^{\infty} \frac{4l}{n^2\pi^2} \cos \frac{n\pi x}{l}.$$

(c) The full Fourier series of $\phi(x) = x$ is

$$\phi(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} (A_n \cos \frac{n\pi x}{l} + B_n \sin \frac{n\pi x}{l})$$

where the coefficients are

$$A_n = \frac{1}{l}\int_{-l}^{l} x\cos(\frac{n\pi x}{l})dx = 0, \quad n = 0, 1, 2, \cdots$$

and

$$\begin{aligned} B_n =& \frac{1}{l}\int_{-l}^{l} x\sin(\frac{n\pi x}{l})dx \\ =& -\frac{x}{n\pi}\cos(\frac{n\pi x}{l})\Big|_{-l}^{l} + \frac{1}{n\pi}\int_{-l}^{l}\cos(\frac{n\pi x}{l})dx \\ =& (-1)^{n+1}\frac{2l}{n\pi}, \quad n = 1, 2, \cdots \end{aligned}$$

Hence

$$x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2l}{n\pi}\sin\frac{n\pi x}{l} = \frac{2l}{\pi}(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} - \cdots).$$

Remark: The full Fourier series and Fourier sine series of x are same, since x is odd.

2. Solve the following problem

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = u(l,t) = 0 \\ u(x,0) = x, \ u_t(x,0) = 0 \end{cases}$$

Solution: By separation of variables, we know that $u(x,t)$ has an expansion

$$u(x,t) = \sum_{n=1}^{\infty}(A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l})\sin\frac{n\pi x}{l}$$

Differentiating with respect to time yields

$$u_t(x,t) = \sum_{n=1}^{\infty}\frac{n\pi c}{l}(-A_n\sin\frac{n\pi ct}{l} + B_n\cos\frac{n\pi ct}{l})\sin\frac{n\pi x}{l}$$

Setting $t = 0$, we have

$$0 = \sum_{n=1}^{\infty}\frac{n\pi c}{l}B_n\sin\frac{n\pi x}{l}$$

so that all the $B_n = 0$. Setting $t = 0$ in the expansion of $u(x,t)$, we have

$$x = \sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{l}$$

By the sine Fourier series of x on the interval $[0,l]$, we know that $A_n = (-1)^{n+1}\frac{2l}{n\pi}, \quad n = 1, 2, \cdots$. Thus

$$u(x,t) = \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}.$$

3. The complex form of the full Fourier series (on P112).

The full Fourier series of $\phi(x)$ is

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n\cos\frac{n\pi x}{l} + B_n\sin\frac{n\pi x}{l}) \tag{1}$$

where the coefficients are

$$A_n = \frac{1}{l}\int_{-l}^{l}\phi(x)\cos(\frac{n\pi x}{l})dx, \quad n = 0, 1, 2, \cdots$$

and

$$B_n = \frac{1}{l}\int_{-l}^{l} \phi(x)\sin(\frac{n\pi x}{l})dx, \quad n = 1, 2, \cdots$$

Note that Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ which implies $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$ and $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, then we should therefore be able to write the full Fourier series in the complex form

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} \tag{2}$$

Multiplying both sides of (2) by $e^{-im\pi x/l}$ and integrating with respect to x yield

$$\int_{-l}^{l} \phi(x)e^{-im\pi x/l} = \sum_{n=-\infty}^{\infty} \int_{-l}^{l} c_n e^{i(n-m)x/l} = 2lc_m$$

where in the second equality we use the following simple fact:

$$\int_{-l}^{l} c_n e^{i(n-m)x/l} dx = \begin{cases} 0, & n \neq m \\ 2lc_m & n = m \end{cases}$$

Hence

$$c_n = \frac{1}{2l}\int_{-l}^{l} \phi(x)e^{-in\pi x/l}dx.$$

Remark: you can check that (1) and (2) are same series written in a different form by using Euler's formula.